# Dynare workshop

## Second and third order approximation

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SUFE, Shanghai, October 28, 2013

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# Computation of first order approximation

- Perturbation approach: recovering a Taylor expansion of the solution function from a Taylor expansion of the original model.
- A first order approximation is nothing else than a standard solution thru linearization.
- A first order approximation in terms of the logarithm of the variables provides standard log-linearization.

# General model

$$E_t \left\{ f(y_{t+1}, y_t, y_{t-1}, u_t) \right\} = 0$$

$$\begin{aligned} E(u_t) &= 0 \\ E(u_t u_t') &= \Sigma_u \\ E(u_t u_\tau') &= 0 \quad t \neq \tau \end{aligned}$$

$y$ : vector of endogenous variables

$u$ : vector of exogenous stochastic shocks

# Timing assumptions

$$E_t \{ f(y_{t+1}, y_t, y_{t-1}, u_t) \} = 0$$

- shocks $u_t$ are observed at the beginning of period $t$,
- decisions affecting the current value of the variables $y_t$, are function of
  - the previous state of the system, $y_{t-1}$,
  - the shocks $u_t$.

# The stochastic scale variable

$$E_t \{f(y_{t+1}, y_t, y_{t-1}, u_t)\} = 0$$

- At period $t$, the only unknown stochastic variable is $y_{t+1}$, and, implicitly, $u_{t+1}$.
- We introduce the *stochastic scale variable*, $\sigma$ and the auxiliary random variable, $\epsilon_t$, such that

$$u_{t+1} = \sigma \epsilon_{t+1}$$

# The stochastic scale variable (continued)

$$E(\epsilon_t) = 0 \tag{1}$$

$$E(\epsilon_t \epsilon_t') = \Sigma_\epsilon \tag{2}$$

$$E(\epsilon_t \epsilon_\tau') = 0 \quad t \neq \tau \tag{3}$$

and

$$\Sigma_u = \sigma^2 \Sigma_\epsilon$$

# Solution function

$$y_t = g(y_{t-1}, u_t, \sigma)$$

where $\sigma$ is the stochastic scale of the model. If $\sigma = 0$, the model is deterministic. For $\sigma > 0$, the model is stochastic.
Under some conditions, the existence of $g()$ function is proven via an implicit function theorem. See H. Jin and K. Judd "Solving Dynamic Stochastic Models"

(http://bucky.stanford.edu/papers/PerturbationMethodRatEx.pdf)

# Solution function (continued)

Then,

$$\begin{aligned} y_{t+1} &= g(y_t, u_{t+1}, \sigma) \\ &= g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma) \\ F(y_{t-1}&, u_t, u_{t+1}, \sigma) \\ &= f(g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t) \end{aligned}$$

$$E_t \left\{ F(y_{t-1}, u_t, \sigma\epsilon_{t+1}, \sigma) \right\} = 0$$

# The perturbation approach

- Obtain a Taylor expansion of the unkown solution function in the neighborhood of a problem that we know how to solve.
- The problem that we know how to solve is the deterministic steady state.
- One obtains the Taylor expansion of the solution for the Taylor expansion of the original problem.
- One consider two different perturbations:
  1. points in the neighborhood from the steady sate,
  2. from a deterministic model towards a stochastic one (by increasing $\sigma$ from a zero value).

# The perturbation approach (continued)

- The Taylor approximation is taken with respect to $y_{t-1}$, $u_t$ and $\sigma$, the arguments of the solution function

$$y_t = g(y_{t-1}, u_t, \sigma).$$

- At the deterministic steady state, all derivatives are deterministic as well.

# Steady state

A deterministic steady state, $\bar{y}$, for the model satisfies

$$f(\bar{y}, \bar{y}, \bar{y}, 0) = 0$$

Furthermore,

$$\bar{y} = g(\bar{y}, 0, 0)$$

# First order approximation

Around $\bar{y}$:

$$
\begin{aligned}
E_t\left\{F^{(1)}(y_{t-1}, u_t, \sigma\epsilon_{t+1}, \sigma)\right\} = &\\
& E_t\Big\{f(\bar{y}, \bar{y}, \bar{y}, 0) + f_{y_+}\left(g_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + g_u\sigma\epsilon' + g_\sigma\sigma\right)\\
& +f_{y_0}\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + f_{y_-}\hat{y} + f_u u\Big\}\\
= \quad & 0
\end{aligned}
$$

with $\hat{y} = y_{t-1} - \bar{y}$, $u = u_t$, $\epsilon' = \epsilon_{t+1}$, $f_{y_+} = \frac{\partial f}{\partial y_{t+1}}$, $f_{y_0} = \frac{\partial f}{\partial y_t}$, $f_{y_-} = \frac{\partial f}{\partial y_{t-1}}$, $f_u = \frac{\partial f}{\partial u_t}$, $g_y = \frac{\partial g}{\partial y_{t-1}}$, $g_u = \frac{\partial g}{\partial u_t}$, $g_\sigma = \frac{\partial g}{\partial \sigma}$.

# Taking the expectation

$$
\begin{aligned}
& E_t\left\{F^{(1)}(y_{t-1},u_t,{\color{red}u_{t+1}},\sigma)\right\} = \\
& \qquad f(\bar{y},\bar{y},\bar{y},0) + f_{y_+}\left(g_y\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+g_\sigma\sigma\right) \\
& \qquad \left.+f_{y_0}\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+f_{y_-}\hat{y}+f_u u\right\} \\
& = \left(f_{y_+}g_y g_y+f_{y_0}g_y+f_{y_-}\right)\hat{y}+\left(f_{y_+}g_y g_u+f_{y_0}g_u+f_u\right)u \\
& \qquad +\left(f_{y_+}g_y g_\sigma+f_{y_0}g_\sigma\right)\sigma \\
& = 0
\end{aligned}
$$

## Solving for $g_y$, $g_u$ and $g_\sigma$

- Dynare uses Klein's approach and the real generalized Schur decomposition.
- This solution verifies Blanchard and Kahn conditions for the existence of a unique stable trajectory.
- Dynare reports an error if these conditions are not satisfied in a given model.
- $g_\sigma = 0$: certainty equilvalence.
- The stable manifold is selected at first order, eliminating explosive roots from the solution.

# First order approximated decision function

$$y_t = \bar{y} + g_y \hat{y} + g_u u$$

$$\begin{aligned} E\left\{y_t\right\} &= \bar{y} \\ \Sigma_y &= g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u' \end{aligned}$$

# Second order approximation of the model

$$
\begin{aligned}
& E_t\left\{F^{(2)}(y_{t-1}, u_t, \epsilon_{t+1}, \sigma)\right\} = \\
& \qquad E_t\Big\{F^{(1)}(y_{t-1}, u_t, u_{t+1}, \sigma) \\
& \qquad +0.5\Big(F_{y\_y\_}(\hat{y}\otimes\hat{y}) + F_{uu}(u\otimes u) + F_{u'u'}\sigma^2(\epsilon'\otimes\epsilon') + F_{\sigma\sigma}\sigma^2\Big) \\
& \qquad +F_{y\_u}(\hat{y}\otimes u) + F_{y\_u'}(\hat{y}\otimes\sigma\epsilon') + F_{y\_\sigma}\hat{y}\sigma + F_{uu'}(u\otimes\sigma\epsilon') + F_{u\sigma}u\sigma + F_{u'\sigma}\sigma\epsilon'\sigma\Big\} \\
= & \qquad E_t\left\{F^{(1)}(y_{t-1}, u_t, \epsilon_{t+1}, \sigma)\right\} \\
& \qquad +0.5\Big(F_{y\_y\_}(\hat{y}\otimes\hat{y}) + F_{uu}(u\otimes u) + F_{u'u'}(\sigma^2\vec{\Sigma}_\epsilon) + F_{\sigma\sigma}\sigma^2\Big) \\
& \qquad +F_{y\_u}(\hat{y}\otimes u) + F_{y\_\sigma}\hat{y}\sigma + F_{u\sigma}u\sigma \\
= & \qquad 0
\end{aligned}
$$

# Representing the second order derivatives

The second order derivatives of a vector of multivariate functions is a three dimensional object. We use the following notation

$$
\frac{\partial^2 F}{\partial x \partial x} = \begin{bmatrix}
\frac{\partial^2 F_1}{\partial x_1 \partial x_1} & \frac{\partial^2 F_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_1}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_1}{\partial x_n \partial x_n} \\
\frac{\partial^2 F_2}{\partial x_1 \partial x_1} & \frac{\partial^2 F_2}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_2}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_2}{\partial x_n \partial x_n} \\
\vdots & \vdots \ddots & \vdots & \ddots & \vdots & \\
\frac{\partial^2 F_m}{\partial x_1 \partial x_1} & \frac{\partial^2 F_m}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_m}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_m}{\partial x_n \partial x_n}
\end{bmatrix}
$$

# Composition of two functions

Let

$$\begin{aligned} y &= g(s) \\ f(y) &= f(g(s)) \end{aligned}$$

then,

$$\frac{\partial^2 f}{\partial s \partial s} = \frac{\partial f}{\partial y} \frac{\partial^2 g}{\partial s \partial s} + \frac{\partial^2 f}{\partial y \partial y} \left( \frac{\partial g}{\partial s} \otimes \frac{\partial g}{\partial s} \right)$$

# Recovering $g_{yy}$

$$
\begin{aligned}
F_{y_- y_-} &= f_{y_+}\left(g_{yy}(g_y \otimes g_y) + g_y g_{yy}\right) + f_{y_0} g_{yy} + B \\
&= 0
\end{aligned}
$$

where $B$ is a term that doesn't contain second order derivatives of $g()$.
The equation can be rearranged:

$$
\left(f_{y_+} g_y + f_{y_0}\right) g_{yy} + f_{y_+} g_{yy}(g_y \otimes g_y) = -B
$$

This is a Sylvester type of equation and must be solved with an appropriate algorithm.

# Recovering $g_{yu}$

$$
\begin{aligned}
F_{y_-u} &= f_{y_+}\left(g_{yy}(g_y \otimes g_u) + g_y g_{yu}\right) + f_{y_0} g_{yu} + B \\
&= 0
\end{aligned}
$$

where $B$ is a term that doesn't contain second order derivatives of $g()$.
This is a standard linear problem:

$$
g_{yu} = -\left(f_{y_+} g_y + f_{y_0}\right)^{-1}\left(B + f_{y_+} g_{yy}(g_y \otimes g_u)\right)
$$

# Recovering $g_{uu}$

$$
\begin{aligned}
F_{uu} &= f_{y_+}\left(g_{yy}(g_u \otimes g_u) + g_y g_{uu}\right) + f_{y_0} g_{uu} + B \\
&= 0
\end{aligned}
$$

where $B$ is a term that doesn't contain second order derivatives of $g()$.
This is a standard linear problem:

$$
g_{uu} = -\left(f_{y_+} g_y + f_{y_0}\right)^{-1}\left(B + f_{y_+} g_{yy}(g_u \otimes g_u)\right)
$$

# Recovering $g_{y\sigma}$, $g_{u\sigma}$

$$
\begin{aligned}
F_{y\sigma} &= f_{y_+} g_y g_{y\sigma} + f_{y_0} g_{y\sigma} \\
&= 0 \\
F_{u\sigma} &= f_{y_+} g_y g_{u\sigma} + f_{y_0} g_{u\sigma} \\
&= 0
\end{aligned}
$$

as $g_\sigma = 0$. Then

$$g_{y\sigma} = g_{u\sigma} = 0$$

# Recovering $g_{\sigma\sigma}$

$$
\begin{aligned}
F_{\sigma\sigma} + F_{u'u'}\Sigma_\epsilon &= f_{y_+}\left(g_{\sigma\sigma} + g_y g_{\sigma\sigma}\right) + f_{y_0} g_{\sigma\sigma} \\
&\quad + \left(f_{y_+y_+}(g_u \otimes g_u) + f_{y_+} g_{uu}\right)\vec{\Sigma}_\epsilon \\
&= 0
\end{aligned}
$$

taking into account $g_\sigma = 0$.
This is a standard linear problem:

$$
g_{\sigma\sigma} = -\left(f_{y_+}(I + g_y) + f_{y_0}\right)^{-1}\left(f_{y_+y_+}(g_u \otimes g_u) + f_{y_+} g_{uu}\right)\vec{\Sigma}_\epsilon
$$

# Second order decision functions

$$y_t = \bar{y} + 0.5 g_{\sigma\sigma}\sigma^2 + g_y\hat{y} + g_u u + 0.5\left(g_{yy}(\hat{y}\otimes\hat{y}) + g_{uu}(u\otimes u)\right) + g_{yu}(\hat{y}\otimes u)$$

We can fix $\sigma = 1$.

Second order accurate moments:

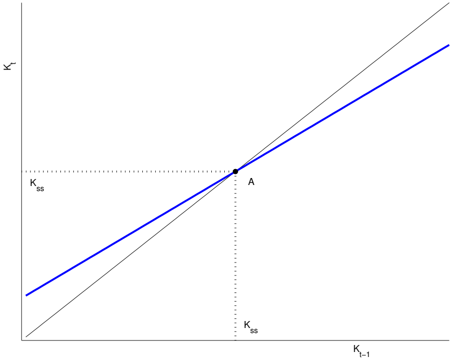
$$\begin{aligned}
\Sigma_y &= g_y\Sigma_y g_y' + \sigma^2 g_u\Sigma_\epsilon g_u' \\
E\left\{y_t\right\} &= \bar{y} + \left(I - g_y\right)^{-1}\left(0.5\left(g_{\sigma\sigma} + g_{yy}\vec{\Sigma}_y + g_{uu}\vec{\Sigma}_\epsilon\right)\right)
\end{aligned}$$

# Three different concepts

1. (deterministic) steady state
2. risky steady state
3. unconditional expectation

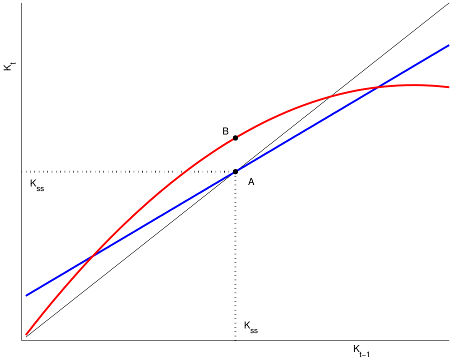
# Deterministic steady state

A linearized decision rule cuts the main diagonal at the deterministic steady state ($K_{ss}$).



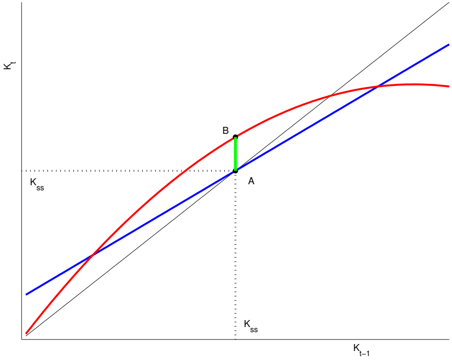
# Quadratic decision rule

In general, the decision is shifted at the deterministic steady state: agents don't decide to stay at the deterministic steady state.



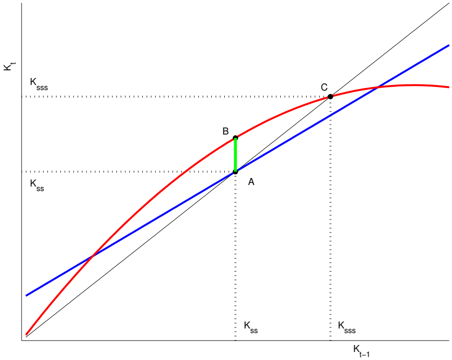
# Quadratic decision rule

The distance between A and B is $g_{\sigma\sigma}/2$



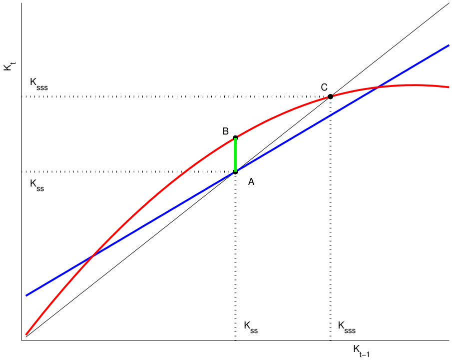
# Risky steady state

The risky steady state, $K_{sss}$, describes the point where agents decide to stay in absence of shocks this period, but taking into account the distibution of shocks in the future.



# Unconditional expectation

Because of Jensen inequality, the unconditional expectation, $E(K)$, is somewhere below the quadratic decision rule, but not on it. In absence of shocks, agents don't decide to go to the unconditional expectation.



# Higher order approximation (I)

The Fa di Bruno formula for the $k$th derivative of the composition of two functions, $f(z(s))$:

$$\left[F^i_{s^j}\right]_{\alpha_1\ldots\alpha_j} = \sum_{l=1}^{j} \left[f^i_{z^l}\right]_{\beta_1\ldots\beta_l} \sum_{c\in\mathcal{M}_{l,j}} \prod_{m=1}^{l} \left[z_{s^{|c_m|}}\right]^{\beta_m}_{\boldsymbol{\alpha}(c_m)}$$

where $\mathcal{M}_{l,j}$ is the set of all partitions of the set of $j$ indices with $l$ classes, $|.|$ is the cardinality of a set, $c_m$ is $m$-th class of partition $c$, and $\boldsymbol{\alpha}(c_m)$ is a sequence of $\alpha$'s indexed by $c_m$. Note that $\mathcal{M}_{1,j} = \{\{1,\ldots,j\}\}$ and $\mathcal{M}_{j,j} = \{\{1\},\{2\},\ldots,\{j\}\}$. In order to keep the formulas compact, we use $\boldsymbol{\alpha}_n$ for $\alpha_1\ldots\alpha_n$.

# Higher order approximation (II)

In order to recover the $k$th order derivatives of the decision function, $g_{y^k}$, it is necessary to solve the following equation:

$$\left(f_{y_+} g_y + f_{y_0}\right) g_{y^k} + f_{y_+} g_{y^k} g_y^{\otimes k} = -B$$

where $g_y^{\otimes k}$ is the $k$th Kronecker power of matrix $g_y$ and $B$ is a term that doesn't contain the unknown $k$-order derivatives of function $g()$, but only lower order derivatives of $g()$ and first to $k$-order derivatives of $f()$.

# Further issues

- Impulse response functions depend of state at time of shocks and history of future shocks.
- For large shocks second order approximation simulation may explode
  - pruning algorithm (Sims)
  - truncate normal distribution (Judd)

# An asset pricing model

Urban Jermann (1998) "Asset pricing in production economies" *Journal of Monetary Economics*, *41*, 257–275.

- real business cycle model
- consumption habits
- investment adjustment costs
- compares return on several securities
- log–linearizes RBC model + log normal formulas for asset pricing

# Firms

The representative firm maximizes its value:

$$\mathcal{E}_t \sum_{t+k}^{\infty} \beta^k \frac{\mu_{t+k}}{\mu_t} D_t$$

with

$$\begin{aligned}
Y_t &= A_t K_{t-1}^{\alpha} \left(X_t N_t\right)^{1-\alpha} \\
D_t &= Y_t - W_t Nt - I_t \\
K_t &= (1-\delta) K_{t-1} + \left( \frac{a_1}{1-\xi} \left( \frac{I_t}{K_{t-1}} \right)^{1-\frac{1}{x}} + a_2 \right) K_{t-1} \\
\log A_t &= \rho \log A_{t-1} + e_t \\
X_t &= (1+g) X_{t-1}
\end{aligned}$$

# Households

The representative households maximizes current value of future utility:

$$\mathcal{E}_t \sum_{k=0}^{\infty} \beta^k \frac{(C_t - \chi C_{t-1})^{1-\tau}}{1-\tau}$$

subject to the following budget constraint:

$$W_t N_t + D_t = C_t$$

and with $N_t = 1$. Good market equilibrium imposes

$$Y_t = C_t + I_t$$

# Interest rate

Risk free interest rate:

$$r_f = \frac{1}{\mathcal{E}_t \left\{ \beta g^{-\tau} \frac{\mu_{t+1}}{\mu_t} \right\}}$$

where $\mu_t$ is the utility of a marginal unit of consumption in period $t$.

$$\mu_t = (c_t - \chi c_{t-1}/g)^{-\tau} - \chi\beta\,(g c_{t+1} - \chi c_t)^{-\tau}$$

# Rate of return

Rate of return of firms

$$
\begin{aligned}
r_t \quad = \quad & \mathcal{E}_t \Bigg\{ a_1 \left( \frac{g i_t}{k_{t-1}} \right)^{-\frac{1}{\xi}} \Bigg( \alpha z_{t+1} g^{1-\alpha} k_t^{\alpha-1} \\
& + \frac{1 - \delta + \frac{a_1}{1-\frac{1}{\xi}} \left( \frac{g i_{t+1}}{k_t} \right)^{1-\frac{1}{\xi}} + a_2}{a_1 \left( \frac{g i_{t+1}}{k_t} \right)^{-\frac{1}{\xi}}} - \frac{g i_{t+1}}{k_t} \Bigg) \Bigg\}
\end{aligned}
$$

# jermann98.mod

```
//----------------------------------------------------------------------
// 1. Variable declaration
//----------------------------------------------------------------------

var c, d, erp1, i, k, r1, rf1, w, y, z, mu;
varexo ez;
```

# (continued)

```
  //-----------------------------------------------------------------------
// 2. Parameter declaration and calibration
//-----------------------------------------------------------------------

parameters alf, chihab, xi, delt, tau, g, rho, a1, a2, betstar, bet;

alf         = 0.36;    // capital share in production function
chihab      = 0.819;   // habit formation parameter
xi          = 1/4.3;   // capital adjustment cost parameter
delt        = 0.025;   // quarterly deprecition rate
g           = 1.005;   //quarterly growth rate (note zero growth =>g=1)
tau         = 5;       // curvature parameter with respect to c
rho         = 0.95;    // AR(1) parameter for technology shock

a1          = (g-1+delt)^(1/xi);
a2          = (g-1+delt)-(((g-1+delt)^(1/xi))/(1-(1/xi)))*
               ((g-1+delt)^(1-(1/xi)));
betstar     = g/1.011138;
bet         = betstar/(g^(1-tau));
```

# (continued)

```
//------------------------------------------------------------------------
// 3. Model declaration
//------------------------------------------------------------------------

model;
g*k  = (1-delt)*k(-1) + ((a1/(1-1/xi))*(g*i/k(-1))^(1-1/xi)+a2)*k(-1);
d    = y - w - i;
w    = (1-alf)*y;
y    = z*g^(-alf)*k(-1)^alf;
c    = w + d;
mu   = (c-chihab*c(-1)/g)^(-tau)-chihab*bet*(c(+1)*g-chihab*c)^(-tau);
mu   = (betstar/g)*mu(+1)*(a1*(g*i/k(-1))^(-1/xi))*(alf*z(+1)*g^(1-alf)*
       (k^(alf-1))+((1-delt+(a1/(1-1/xi))*(g*i(+1)/k)^(1-1/xi)+a2))/
       (a1*(g*i(+1)/k)^(-1/xi))-g*i(+1)/k);
log(z) = rho*log(z(-1)) + ez;
```

# (continued)

```
rf1  = 1/expecation(0)(betstar/g)*mu(+1)/mu);
r1   = (a1*(g*i/k(-1))^(-1/xi))*(alf*z(+1)*g^(1-alf)*(k^(alf-1))+
       (1-delt+(a1/(1-1/xi))*(g*i(+1)/k)^(1-1/xi)+a2)/
       (a1*(g*i(+1)/k)^(-1/xi))-g*i(+1)/k);
erp1 = r1 - rf1;

end;
```

# (continued)

```
steady_state_model;
rf1    = (g/betstar);
r1     = (g/betstar);
erp1   = r1-rf1;
z      = 1;
k      = (((g/betstar)-(1-delt))/(alf*g^(1-alf)))^(1/(alf-1));
y      = (g^(1-alf))*k^alf;
w      = (1-alf)*y;
i      = (1-(1/g)*(1-delt))*k;
d      = y - w - i;
c      = w + d;
mu     = ((c-(chihab*c/g))^(-tau))-chihab*bet*((c*g-chihab*c)^(-tau));
ez     = 0;
end;
```

# (continued)

```
steady;

shocks;
var ez; stderr 0.01;
end;

stoch_simul (order=2) rf1, r1, erp1, y, z, c, d, mu, k;
```

# 3rd order approximation

- same principle of derivation as 2nd order
- Don't forget options periods= in order to compute empirical moments
- No pruning at 3rd order